Lecture 3: Iterative methods for tensors

Whitening:

Given: \( M = \sum \lambda_i \, u_i u_i^T \) for \( u_i \)'s non-orthogonal
\[ T = \sum \lambda_i \, u_i \]

Let \( V D V^T \) be eigendecomposition of \( M \), define
\[ W = V D^{-1/2} \in \mathbb{R}^{d \times k} \]
\[ u_i = \lambda_i \, W^T u_i \in \mathbb{R}^k \]

Motivation: \( W \) transforms \( M \) into \( \mathbb{I}_{d_k} \):
\[
WMW = D^{-1/2} V^T V D V^T V D^{-1/2} = D^{-1/2} D D^{-1/2} = \mathbb{I}_{d_k}
\]
\[
= \sum \lambda_i (W^T u_i)(W^T u_i) = \sum \hat{u}_i \hat{u}_i^T
\]
So \( \hat{u}_i \)'s are orthonormal basis for \( \mathbb{R}^k \).
Then consider

\[ T' \equiv T(W, W, W) \in \mathbb{R}^{k \times k \times k} \]

where \( T'(x, y, z) \equiv T(Wx, Wy, Wz) \)

Then

\[ T'(x, y, z) = \sum_{i} \lambda_i \langle Wx, u_i \rangle \langle Wy, u_i \rangle \langle Wz, u_i \rangle \]

\[ = \sum_{i} (x^T W^T u_i) \cdot (y^T W^T u_i) \cdot (z^T W^T u_i) \]

\[ = \sum_{i} \lambda_i (x_i^{1/2} u_i, x) \cdot (y_i^{1/2} u_i, y) \cdot (z_i^{1/2} u_i, z) \]

\[ = \sum_{i} \lambda_i^{-1/2} \langle u_i, x \rangle \langle u_i, y \rangle \langle u_i, z \rangle \]

So

\[ T' = \sum_{i} \lambda_i^{1/2} u_i \otimes 3 \]

i.e. \( T' \) has orthogonal components.
Tensor power method for non-orthogonal components:

(without whitening) [Vatsal-Sharan '17]

\[ T = \sum_{i=1}^{k} u_i \otimes 3 \]

Iterates of tensor power method \((Z_t)\) given by:

\[ Z'_{t} = T(Z_{t-1}, Z_{t-1}, \ldots) \]

\[ Z_t = Z'_t / \|Z'_t\| \]

Define \(a_{i,t} \equiv \langle u_i, Z_t \rangle\), \(\hat{a}_{i,t} \equiv \frac{a_{i,t}}{a_{1,t}}\)

\[ a_{j,t} = \frac{\sum_i a_{i,t-1}^2 \langle u_i, u_j \rangle}{\|Z'_t\|} = \frac{\hat{a}_{1,t-1} \sum_i a_{i,t-1}^2 c_{i,j}}{\|Z'_t\|} \]

Because \(\|Z'_t\|\) is fixed factor independent of \(j\),

\[ a_{j,t} = \frac{\sum_i a_{i,t-1}^2 c_{i,j}}{\sum_i a_{i,t-1}^2 c_{i,1}} \]
Rewriting this further to isolate \( i=1 \) terms,

\[
\hat{\alpha}_{j,t} = \frac{c_{i,j} + \sum_{i+1} \hat{a}_{i,t-1}^2 c_{i,j}}{1 + \sum_{i+1} \hat{a}_{i,t-1}^2 c_{i,j}} \quad \text{for } |1| \leq k c_{\text{max}} << 1
\]

\[
\leq \left( c_{i,j} + \sum_{i+1} \hat{a}_{i,t-1}^2 c_{i,j} \right) \left( 1 - \sum_{i+1} \hat{a}_{i,t-1}^2 c_{i,j} \right) \quad (1)
\]

We show

\[
\max_{j+1} |\hat{\alpha}_{j,t}| < \beta_t \quad (6)
\]

for sequence \((\beta_t)\) defined recursively by

\[
\beta_0 = \max_{j+1} |\hat{\alpha}_{j,0}|
\]

\[
\beta_t = c_{\text{max}} + \beta_{t-1}^2 + 3 k c_{\text{max}} \beta_{t-1}^2
\]

Provided \( k c_{\text{max}} << 1 - \beta_0 \), can show \( \beta_t \leq 1 \) (proof omitted)

**Proof of (6):**

Note

\[
|c_{i,j} + \sum_{i+1} \hat{a}_{i,t-1}^2 c_{i,j}| = |c_{i,j} + \alpha_{j,t-1} + \sum_{i+1} \hat{a}_{i,t-1} c_{i,j}|
\]

\[
\leq c_{\text{max}} + \beta_{t-1}^2 + k c_{\text{max}} \beta_{t-1}^2
\]
So by (1),

\[ |\hat{a}_{i, t}^j| < (c_{\text{max}} + \beta_{t-1}^2 + k c_{\text{max}} \beta_{t-1}^2) (1 + k c_{\text{max}} \beta_{t-1}^2) \]

\[ \leq c_{\text{max}} + \beta_{t-1}^2 + 2 k c_{\text{max}} \beta_{t-1}^2 < \beta_t. \]

Then suffices to analyze the recursion defining \( \beta_t \) (note: we have reduced keeping track of \( k-1 \) different quantities \( \hat{a}_{2, t}, \ldots, \hat{a}_{k, t} \) to just keeping track of a single quantity!).

To get intuition, consider case where \( A_i \)'s orthogonal, i.e. \( c_{\text{max}} = 0 \).

Then \( \beta_+ = \beta_{t-1}^2 \), so

\[ \beta_+ = \beta_0^2 \]

i.e. if \( \beta_0 < 1 \), then \( \beta_t \to 0 \) at doubly exponential rate.
For $c_{\text{max}} > 0$ case, $\beta_0$ has to be sufficiently smaller than 1, i.e.

$$1 - \beta_0 \gg k c_{\text{max}}$$

If $c_{\text{max}} < \frac{1}{k^2}$, then this is satisfied w.h.p by randomly initializing (proof omitted, see Lemma 1 in [Sharan-Valiant], link on course page).

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Analyzing recursion for $(\beta_t)$:

3 stages:

1. $\beta_t \geq 0.1$

2. $0.1 \geq \beta_t \geq \sqrt{\gamma} \quad \gamma \equiv \max(c_{\text{max}}, 1/d)$

3. $\beta_t \leq \sqrt{\gamma} \leftarrow \smiley$

Stage 1: note $c_{\text{max}} \leq k c_{\text{max}} \beta_t^{2-1}$

b/c $k \beta_t^{2-1} \geq 0.1 k \geq 1$

(for $k$ larger than some constant)
\[ \beta_t \leq (1+4k\max)^2 \beta_{t-1}, \]

and unrolling, we get

\[ \beta_t \leq (1+4k\max)^{1+2^{t-1}} \beta_0 \]

\[ = (1+4k\max)^{2^{t-1}} \beta_0^2 \]

\[ \leq \left[ \beta_0 (1+4k\max) \right]^{2^t} \]

so if \( \beta_0 \leq 1-5k\max \) (which happens w.h.p.),

\[ \leq \left( 1 - \frac{k\max}{k^2} \right)^{2^t} \]

so we stay in this stage for \( \lg k \) iterations.

Stage 2: Reindex so \( \beta_0 \) is start of this stage

because \( \beta_t \geq \sqrt{3} \geq \sqrt{\max} \),

\[ \beta_t = (1+3k\max)\beta_{t-1} + \max \leq (2+3k\max)\beta_{t-1} \leq 3 \beta_{t-1} \]
unrolling, we get

\[ \beta_+ \leq 3^{2^t - 1} \beta_0 \]

\[ \leq (3 \beta_0)^{2^t} \]

\[ \leq (0.3)^{2^t} \]

So we stay in this stage for \( O(\lg \lg n) \) iterations. \( \square \)